LARGE MATCHINGS WITH FEW COLORS

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Abstract. Let K_n^r denote the complete r-uniform hypergraph on n vertices. A matching M in a hypergraph is a set of pairwise vertex disjoint edges, and an s-colored matching is a matching using edges from at most s colors. Recent Ramsey-type results rely on lemmas about the size of monochromatic matchings in hypergraphs. A starting point for this study comes from a well-known result of Alon, Frankl, and Lovász 1, which states that any edge-coloring of the complete r-uniform hypergraph on n vertices with t colors contains a monochromatic matching of size k. A natural extension of this theorem would find the smallest n such that every t-coloring of K_n^r contains an s-colored matching of size k. It has been conjectured that in every coloring of the edges of K_n^r with 3 colors there is a 2-colored matching of size at least k provided that $n \geq kr + \left\lfloor \frac{k-1}{r+1} \right\rfloor$. The smallest non-trivial case is when $r = 3$ and $k = 4$. We prove that in every 3-coloring of the edges of K_{12}^3 there is a 2-colored matching of size 4.

Introduction

Throughout this paper we will rely on the following result of Alon, Frankl, and Lovász[1], solving a conjecture of Erdős from 1978[2].

Theorem 1 ([1]). Suppose that $n = (t-1)(k-1) + kr$. Then any edge-coloring with t colors of the complete r-uniform hypergraph on n vertices, K_n^r, induces a monochromatic matching of size at least k.

Extending this idea a bit further, Gyárfás, Sárközy, and Selkow[3] introduced an extra parameter $s \in \{1, 2, \ldots, t\}$. A matching in a hypergraph which uses edges from at most s colors is called an *s-colored matching*. It is natural to ask what is the smallest n such that every t-edge coloring of K_n^r contains a s-colored matching of size k? [4] states the following conjecture for certain values of $1 \leq s \leq t$ (in particular for $t = 3,\ s = 2$).

Conjecture 2 ([4]). Every t-coloring of the edges of K_n^r (a *t-edge-coloring*) contains an s-colored matching of size k provided that $n \geq kr + \left\lfloor \frac{(k-1)(t-s)}{1+r+r^2+\cdots r^{s-1}} \right\rfloor$. (We also note here that the case $s = 1$ is 1; the case $s = t$ is trivial.)

The case of $s = t-1,\ r = 2$ is solved in [3]. The smallest open case of 2 is $t = 3$, $s = 2$, $r = 3$, $k = 4$. With this in mind, our main result is the following:

Theorem 3. Every 3-edge-coloring of the complete 3-uniform hypergraph on 12 vertices, K_{12}^3, contains a perfect 2-colored matching.

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Proof of Main Result

Throughout the following, we assume that $\mathcal{H}$ is a 3-edge-colored K_{12}^3 without a 2-colored matching of size 4.

Definition 4. A set of 6 vertices from $\mathcal{H}$ is called a *B-set* if one color is avoided by all edges induced by these vertices and no disjoint pair of edges is monochromatic.

Definition 5. Call a set of 7 vertices a B^+ set if every 6-vertex subset chosen out of it is a B-set avoiding the same color.

Lemma 6. $\mathcal{H}$ contains no B^+ set.

Proof of Lemma 6. Suppose that we have a B^+-set with vertex-set $\{v_1, v_2, \dots, v_7\}$ with $\binom{B}{[3]}$ colored c_1 and c_2. Assume without loss of generality that $\{v_1, v_2, v_3\}$ is c_1, and so $\{v_4, v_5, v_6\}$ must be c_2. Replace one vertex of one of the two matched edges by the left-over vertex. Then $\{v_5, v_6, v_7\}$ is c_2, $\{v_2, v_3, v_4\}$ is c_1, $\{v_6, v_7, v_1\}$ is c_2, and so $\{v_3, v_4, v_5\}$ is also c_1, a contradiction. $\square$

Definition 7. Call $A \subseteq V(H)$, $|A| = 6$ an *A-set*, if A contains a monochromatic perfect matching.

Lemma 8. The complement of an A-set is a B-set.

Proof of Lemma 8. Let A be an A-set containing a monochromatic perfect matching M_1 of color c_1. Observe that if the complement $B := V(\mathcal{H}) \setminus A$ is an A-set, then $\mathcal{H}$ contains a 2-colored perfect matching consisting of M_1 and the monochromatic perfect matching in B. So any perfect matching in B is 2-colored, and does not contain the color c_1, otherwise again it would induce together with M_1 a 2-colored perfect matching in $\mathcal{H}$. This implies that B is a B-set in colors c_2 and c_3. $\square$

Lemma 9. In any balanced partition of $V(\mathcal{H}) = Y \dot\cup Z$, at least one of the partition classes is a B-set.

Proof of Lemma 9. We may assume that neither Y nor Z is an A-set, otherwise by Lemma 8 the complement is a B-set. Suppose that Y is not a B-set. Then there is a 2-colored matching M_1 in Y, say in c_1 and c_2, and another 2-colored matching M_2 in Y, say in c_2 and c_3. If Z has a matching M_3 with the same colors as M_1 or as M_2, the matchings $M = M_3 \dot\cup M_1$, or $M = M_3 \dot\cup M_2$, respectively, induces a 2-colored perfect matching in $\mathcal{H}$. This implies that the edges of any matching M_3 in Z have two distinct colors, c_1 and c_3. Thus Z is a B-set. $\square$

Lemma 10. $\mathcal{H}$ contains no K_5^3 using all of $\{c_1, c_2, c_3\}$.

Proof of Lemma 10. Suppose that there is a K_5^3 using colors c_1, c_2, c_3 spanning a vertex set $X \subseteq V(\mathcal{H})$. Take an arbitrary vertex $v \in V(\mathcal{H}) \setminus X =: Y$. Then the set $X \cup \{v\}$ is not a B-set, implying, by Lemma 9, that $Y \setminus \{v\}$ is a B-set. As the choice of vertex v was arbitrary, the same holds for any vertex $v \in Y$. We label the vertices of Y by pairs of colors in the following way: a vertex $v \in Y$ is assigned label $\chi(v) = \{c_1, c_2\}$ if the triples in the B-set, $Y \setminus \{v\}$, are colored by c_1 or c_2.

First we show that only two of the three possible color pairs may occur. To see this, suppose we have vertices $v_1, v_2, v_3 \in Y$ with $\chi(v_1) = \{c_1, c_2\}$, $\chi(v_2) = \{c_1, c_3\}$, and $\chi(v_3) = \{c_2, c_3\}$. As $|Y \setminus \{v_1, v_2, v_3\}| = 4$ there is an edge in $Y \setminus \{v_1, v_2, v_3\}$ and its color should be either c_1 or c_2, and either c_1 or c_3, and either c_2 or c_3, a contradiction.

Now, assume, without loss of generality, that there is a set $U \subseteq Y$ with vertices labeled $\{c_1, c_2\}$ and a set $W \subseteq Y$ with vertices labeled $\{c_1, c_3\}$, and that $|U| > |W|$. First assume that $W = \emptyset$. In this case $Y = U$ is a B^+-set, a contradiction by Lemma 6. Thus $W \neq \emptyset$, so pick a vertex $w \in W$. Then in any partition of $Y \setminus \{w\}$ into two disjoint edges, there is one edge, say e, that is colored c_3. As $|U| \geq 4$ there is a vertex $u \in U \setminus e$. By definition, any edge in $Y \setminus \{u\}$ is colored either c_1 or c_2, a contradiction with the fact that e is c_3. Thus there is no K_5^3 in $\mathcal{H}$ using all three colors.

$\square$

Lemma 11. For any pair of vertices $u, v \in V(\mathcal{H})$ there exists a monochromatic $K_6^3 \subset \mathcal{H}$ containing u and v.

Proof of Lemma 11. Considering any pair of vertices $x, y \in \mathcal{H}$, we know from Lemma 10 that x and y are not contained in any K_5^3 using all of $\{c_1, c_2, c_3\}$. Thus the edges containing x and y are of at most two colors. As in the proof of Lemma 10, label the pair $\{x, y\}$ by the pair of colors it induces. If all edges containing x and y are of the same color, then choose an arbitrary color to complete the pair. This defines a coloring of the edges of the complete graph K_{12}^2.

Let us choose two arbitrary vertices $u, v \in \mathcal{H}$. By Theorem 1 there is a monochromatic matching M of size 3 consisting of the 2-edges $\{e_1, e_2, e_3\}$ in the graph induced by the ten vertices $V(\mathcal{H}) \setminus \{u, v\}$, K_{10}^2. Suppose that M is colored by the color-pair $\{c_1, c_2\}$. We claim that $V(\mathcal{H}) \setminus V(M)$ induces a monochromatic K_6^3 colored in c_3 in the original hypergraph. If this were not the case, there would exist a 3-edge E of $\mathcal{H}$ with color either c_1 or c_2. Let $\{v_1, v_2, v_3\}$ be the set $V(\mathcal{H}) \setminus (V(M) \cup E)$. Then the perfect matching $\{E, e_1 \cup v_1, e_2 \cup v_2, e_3 \cup v_3\}$ would not contain the color c_3, contradicting our assumption on $\mathcal{H}$. Thus, u and v are contained in the monochromatic K_6^3 colored in c_3 on the vertex set $V(\mathcal{H}) \setminus V(M)$.

□

Lemma 12. Any two monochromatic $K_6^3 \subset \mathcal{H}$ intersecting in at most 4 vertices have different colors.

Proof of Lemma 12. Consider two distinct monochromatic $K_6^3 \subset \mathcal{H}$, X_1, X_2, of the same color, say c_1, with $|V(X_1) \cap V(X_2)| \leq 4$. If $|V(X_1) \cap V(X_2)| \leq 3$, then they span at least 9 vertices and we would have 3 disjoint edges of the same color. Therefore, we may assume that $|V(X_1) \cap V(X_2)| = 4$. Consider then the 4 vertices not contained in either $V(X_1)$ or $V(X_2)$, $\bar{X} = V(\mathcal{H}) \setminus \big(V(X_1) \cup V(X_2)\big)$. Observe that no edge e with $|e \cap \bar{X}| = 1$ is c_1, otherwise we would have 9 vertices forming three disjoint edges colored in c_1. Therefore any such e must be colored either c_2 or c_3, and as $|\bar{X}| = 4$ they induce a perfect matching colored only in c_2 and c_3 which is forbidden in $\mathcal{H}$. Thus the two distinct monochromatic K_6^3 must have different colors.

□

Lemma 13. Any two distinct monochromatic K_6^3 of different colors intersect in exactly 2 vertices.

Proof of Lemma 13. First, observe that trivially two monochromatic K_6^3, of different colors, say a c_1, X_1 and a c_2, X_2 cannot meet in more than 2 vertices. Otherwise any edge contained in their intersection must be of both colors. Also, X_1 and X_2 must intersect, otherwise $V(X_1) \cup V(X_2)$ induces a 2-colored perfect matching. This produces a contradiction with the structure of $\mathcal{H}$

It remains to show that X_1 and X_2 cannot intersect in only one vertex. If this were the case, then by Lemma 11 any vertex $v \in V(\mathcal{H}) \setminus \big(V(X_1) \cup V(X_2)\big)$ is covered by a monochromatic K_6^3, X. As $|V(X) \cap \big(V(X_1) \cup V(X_2)\big)| = 5$, the complete hypergraph X intersects X_1 or X_2 in at least 3 vertices, hence its color is either c_1 or c_2, say c_1. Then $|X \cap X_1| \geq 5$ by Lemma 12. We choose two edges colored c_2 from X_2, and an edge colored c_1 containing v from X. One can easily see that the remaining vertices are in X_1, hence the edge containing them is c_1. This yields a 2-colored matching, which is a contradiction.

□

Definition 14. Call an edge-colored 3-uniform hypergraph a *disk* if it contains three monochromatic complete subgraphs on six vertices, X_1, X_2, X_3, each colored a different color such that $|V(X_i) \cap V(X_j)| = 2$ for $i, j = 1, 2, 3$ and $V(X_1) \cap V(X_2) \cap V(X_3) = \emptyset$.

Lemma 15. $\mathcal{H}$ is a disk.

Proof of Lemma 15. Choose any pair of vertices $u_1, v_1 \in V(\mathcal{H})$. By Lemma 11, there is a monochromatic complete hypergraph X_1 of size six covering u_1 and v_1. Choose a pair of vertices $u_2, v_2 \in V(\mathcal{H}) \setminus V(X_1)$. By Lemma 11, there is a monochromatic complete hypergraph X_2 covering u_2, v_2. By Lemma 12, the complete hypergraphs X_1 and X_2 have different colors, and by Lemma 13, we have $|V(X_1) \cap V(X_2)| = 2$. Thus $|V(\mathcal{H}) \setminus (V(X_1) \cup V(X_2))| = 2$. Let u_3, v_3 be the two vertices from $V(\mathcal{H}) \setminus (V(X_1) \cup V(X_2))$. Again, by Lemma 11, there is a monochromatic complete hypergraph X_3 covering u_3, v_3. By Lemma 12, the color of the edges of X_3 is different from the color of X_1 and of X_2. By Lemma 13, the hypergraphs X_3 and X_i, $i = 1, 2$, intersect in exactly two vertices, hence $\mathcal{H}$ is a disk. $\square$

Lemma 16. Any disk, $\mathcal{D}$, contains a K_5^3 using all three colors.

Proof of Lemma 16. Since $\mathcal{D}$ is a disk, it contains X_1, X_2 and X_3 colored c_1, c_2 and c_3, respectively. Let $v_1, v_2 \in V(X_1) \cap V(X_2)$, $v_3, v_4 \in V(X_2) \cap V(X_3)$ and $v_5 \in V(X_1) \cap V(X_3)$. Then the edge $\{v_1, v_2, v_5\}$ is of color c_1, the edge $\{v_1, v_2, v_3\}$ is of color c_2, and the edge $\{v_3, v_4, v_5\}$ is of color c_3. Thus the complete 3-uniform hypergraph induced on $\{v_1, \ldots, v_5\}$ uses all three colors. $\square$

Proof of Theorem 3. Let $\mathcal{H}$ be any edge-colored K_{12}^3 which does not contain a perfect matching using at most two colors. By Lemma 15, $\mathcal{H}$ is a disk. Thus by Lemma 16 $\mathcal{H}$ contains a K_5^3 using all three colors. This is a contradiction to Lemmas 10, thus no such H exists. This proves Theorem 3 $\square$

Theorem 17. In every edge-coloring of a 3-uniform hypergraph K_{16}^3 on 16 vertices in three colors, there is a 2-colored matching of size 5.

Proof of Theorem 17. Let χ be a given edge-coloring of $\mathcal{H} = K_{16}^3$ by 3 colors. Set $r = 3$, $t = 3$ and $k = 3$. Then $16 \geq 13 = (t-1)(k-1) + kr$ implying $\mathcal{H}$ contains matching M_1 of size 3 that is monochromatic, say in c_1. If the induced subgraph$\mathcal{H}'$ on $V(\mathcal{H}) \setminus V(M_1)$ contains a c_1 edge e, then $M_1 \cup e$ together with any edge from the remaining 4 vertices form a matching of size 5 colored with two colors, and we are done. So we may assume that there is no c_1-colored edge in $\mathcal{H}'$.

By Lemma 6, $V(\mathcal{H}')$ is not a B^+-set and thus contains a monochromatic matching M_2. Then $M = M_1 \cup M_2$ leads to the desired 2-colored matching of size 5. $\square$

Theorem 18. In every 3-edge-coloring of a 3-uniform hypergraph K_{19}^3 on 19 vertices, there is a 2-colored matching of size 6.

Proof of Theorem 18. This proof follows the proof of Theorem 17. Theorem 1 leads to a monochromatic matching M of size 4 and in the reaming 7 vertices either we find an edge of the same color as M or a monochromatic pair of edges. $\square$

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